Towards a General Theory of Stochastic Hybrid Systems

Manuela L. Bujorianu* John Lygeros† Marius C. Bujorianu

March 12, 2008

Abstract

In this chapter we set up a mathematical structure, called Markov string, to obtaining a very general class of models for stochastic hybrid systems. Markov Strings are, in fact, a class of Markov processes, obtained by a mixing mechanism of stochastic processes, introduced by Meyer. We prove that Markov strings are strong Markov processes with the cadlag property. We then show how a very general class of stochastic hybrid processes can be embedded in the framework of Markov strings. This class, which is referred to as the General Stochastic Hybrid Systems (GSHS), includes as special cases all the classes of stochastic hybrid processes, proposed in the literature.

Keywords: stochastic hybrid systems, Markov string, Markov processes, strong Markov property, cadlag, generator.

1 Introduction

In the face of growing complexity of control systems, stochastic modelling has got a crucial role. Indeed, stochastic techniques for modelling control and hybrid systems have attracted attention of many researchers and constitute one of the hottest issues in contemporary high level research.

Hybrid systems have been extensively studied in the past decade, both concerning their theoretical framework, as well as relating to the increasing number of applications they are employed for. However, the subfield of stochastic hybrid systems is fairly young. There has been considerable current interest in stochastic hybrid systems due to their ability to represent such systems as maneuvering aircraft [HHT03], switching communication networks [Hes04]. Different issues related to stochastic hybrid systems have found applications to insurance pricing [DV95], capacity expansion models for the power industry [DDSV87], flexible manufacturing and fault tolerant control [GAM93, GAM97], etc.

A considerable amount of research has been directed towards this topic, both in the direction of extending the theory of deterministic hybrid systems [HLS00], as well as discovering new applications unique to the probabilistic framework.

This paper has three objectives:

1. Introduce a very general framework for modelling stochastic hybrid processes: General Stochastic Hybrid System, abbreviated with GSHS.

2. Develop a theoretical construction for mixing Markov processes which preserves the Markov property. The result of this mixing operation will be called *Markov string*.

*Department of Computer Science, University of Twente, Enschede, *email:l.m.bujorianu@cs.twente.nl*

†Automatic Control Laboratory, ETH Zurich, ETLI 22, Switzerland, *email: lygeros@control.ee.ethz.ch*

3. Show how GSHS can be embedded in the Markov string constructions and hence deduce the basic properties of GSHS as Markov property, strong Markov property

A GSHS might be thought of a ‘conventional’ hybrid system enriched with three uncertainty characteristics:

1. the continuous-time dynamics are driven by stochastic differential equations (SDE) rather then classical ODE,

2. a jump takes place when the continuous state hits the mode boundary or according with a transition rate

3. the post jump locations are randomly chosen according with a stochastic kernel.

Intuitively, GSHS can be described as an interleaving between a finite or countable family of diffusion processes and a jump process. Our goal is to prove that GSHS is indeed a ‘good model’. This means that we need to investigate the stochastic properties of this model. A natural property we were looking for is the Markov property. Analysing the form of the GSHS executions (paths or trajectories), the first observation is that these are, in fact, ‘concatenations’ of the diffusion component paths. The continuity inherited from the diffusion trajectories is perturbed by the jumps between the diffusion components.

This observation leads to the investigation of a general mechanism for mixing Markov processes that preserves the Markov property. Given a finite or countable family of Markov processes with reasonably good properties, this machinery will allow us to get a new Markov process whose paths are obtained by ‘sticking’ together the component paths. Roughly speaking, Markov strings are sequences of Markov processes. The jump structure of a Markov string is completely described by a renewal kernel given a priori and a family of terminal times associated with the initial processes. We require that the Markov string have finitely many jumps in finite time. Under these assumptions we prove that the Markov strings, as stochastic processes, enjoy useful properties like the strong Markov property and the càdlàg property.

We then return to GSHS and show how GSHS can be embedded in the framework of Markov strings. The class of GSHS inherits the strong Markov and càdlàg properties from Markov strings.

Finally, we develop the expression of the infinitesimal generator associated to GSHS.

2 Motivation from Air Traffic Control

The ultimate goal of our work (under the European Commission’s HYBRIDGE project [HYB]) is to use theoretical tools developed for stochastic hybrid models as a basis for designing and analyzing advanced Air Traffic Management (ATM) concepts for the European airspace. The modelling of ATM systems is a stochastic hybrid process, since it involves the interaction of continuous dynamics (e.g. the movement of the aircraft), discrete dynamics (e.g. aircraft landing and taking off, moving from one air traffic control sector to another, etc.) and stochastic dynamics (e.g. due to wind, uncertainty about the actions of the human operators, malfunctions, etc.).

In the context of ATM we are interested in modelling and analysing safety-critical situations. In [WL], a number of such situations were identified. Each one appears to have different modelling needs. In the following, we highlight the stochastic hybrid issues that arise in two aspects of ATM modelling: aircraft and weather models. Different models developed in the literature for stochastic hybrid processes might be used to model different safety critical situations identified in ATM. The difference between these models consists in where the stochastic phenomena appear: in the discrete dynamics, in the continuous dynamics or in both. For different safety-critical situations identified in the ATM modelling different models might be appropriate depending where the randomness lies:

- In the modelling of *aircraft climbing* the most suitable models appear to be SHS [HLS00].
- Uncertainty in the ATC *sector transition process* can be treated in the framework of PDMP [BL03].
- For *missed approaches*, an appropriate model seems to be the SDP model [GAM97]. SDP can also model changes in the flight plan segment when the aircraft reaches a way point (by introducing rate functions with support in a neighborhood of the way point). For missed approaches due to runway incursions, a general stochastic hybrid systems model is needed to accurately model this case.
- For modelling *overtake manoeuvres* in unmanaged airspace the most appropriate models are SDP [GAM97].

For more details see [BLGP02]. The conclusions of the above discussion is that it is necessary to develop further a more general class of stochastic hybrid processes than those found in the literature. This is because

1. Different types of models seem to be needed to capture the different situations. This implies that a number of different techniques and tools must be mastered to be able to deal with all the cases of interest. If a GSHS framework were available the process would be more efficient, since a single set of results, simulation procedures, etc. could be used in all cases.
2. Certain situations, such as vertical crossings during descent and missed approaches due to runway incursions, would be more accurately modelled by a GSHS.

3 Background on Markov Processes

In the following we make use of some standard notions from the Markov process theory as: underlying probability space, natural filtration, translation operator, Wiener probabilities, admissible filtration, stopping time, strong Markov property [BG68]. The basic definitions from the Markov process theory are summarized below[1].

Suppose that $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x), \in Q$ is a Markov process. We denote the state space of $\mathbb{M}$ by $(X, \mathcal{B})$ and assume that $\mathcal{B}$ is the Borel σ-algebra of X if X is a topological Hausdorff space. Let Δ be the cemetery point for X, which is an adjoined point to X, $X_\Delta = X \cup \{\Delta\}$. The existence of Δ is assumed in order to have a probabilistic interpretation of $P_x(x_t \in X) < 1$, i.e. at some 'termination time' $\zeta(\omega)$ when the process $\mathbb{M}$ escapes to and is trapped at Δ. The elements $\mathcal{F}$, $\mathcal{F}_t^0$, $\mathcal{F}_t$, θ_t, P, P_x have the usual meaning, i.e.

- $(\Omega, \mathcal{F}, P)$ denotes the underlying probability space.
- $\mathcal{F}_t^0$ denotes the *natural filtration*, i.e. $\mathcal{F}_t^0 = \sigma\{x_t, s \leq t\}$ and $\mathcal{F}_\infty^0 = \vee_t \mathcal{F}_t^0$.

[1] this section could be missing in the final version

- $x_t : (\Omega, \mathcal{F}) \to (X, \mathcal{B})$ is a $\mathcal{F}^0/\mathcal{B}$-measurable function for all $t \geq 0$.
- $\theta_t : \Omega \to \Omega$, for all $t \geq 0$, is the *translation operator*, i.e.

$$x_s \circ \theta_t = x_{t+s},\ t, s \geq 0$$

- $P_x : (\Omega, \mathcal{F}^0) \to [0, 1]$ is a probability measure (so-called *Wiener probability*) such that $P_x(x_t \in E)$ is $\mathcal{B}$-measurable in $x \in X$ for each $t \geq 0$ and $E \in \mathcal{B}$.
- If $\mu \in \mathcal{P}(X_\Delta)$, i.e. μ is a probability measure on $(X, \mathcal{B})$ then we can define

$$P_\mu(\Lambda) = \int_{X_\Delta} P_x(\Lambda)\mu(dx), \Lambda \in \mathcal{F}^0.$$

  We then denote by $\mathcal{F}$ (resp. $\mathcal{F}_t$) the completion of $\mathcal{F}^0_\infty$ (resp. $\mathcal{F}^0_t$) with respect to all P_μ, $\mu \in \mathcal{P}(X_\Delta)$.
- We say that a family $\{\mathcal{M}_t\}$ of sub-σ-algebras of $\mathcal{F}$ is an admissible filtration if $\mathcal{M}_t$ is increasing in t and $x_t \in \mathcal{M}_t/\mathcal{B}$ for each $t \geq 0$. Then $\mathcal{F}^0_t$ is the *minimum admissible filtration*. An admissible filtration $\{\mathcal{M}_t\}$ is *right continuous* if $\mathcal{M}_t = \mathcal{M}_{t+} = \cap\{\mathcal{M}_{t'} | t' > t\}$.
- Given an admissible filtration $\{\mathcal{M}_t\}$, a $[0, \infty]$-valued function τ on Ω is called an $\{\mathcal{M}_t\}$-*stopping time* if $\{\tau \leq t\} \in \mathcal{M}_t$, $\forall t \geq 0$.
- For an admissible filtration $\{\mathcal{M}_t\}$, we say that $\mathbb{M}$ is *strong Markov* with respect to $\{\mathcal{M}_t\}$ if $\{\mathcal{M}_t\}$ is right continuous and

$$P_\mu(x_{\tau+t} \in E | \mathcal{M}_\tau) = P_{x_\tau}(x_t \in E);\ P_\mu - a.s.$$

  $\mu \in \mathcal{P}(X_\Delta)$, $E \in \mathcal{B}$, $t \geq 0$, for any $\{\mathcal{M}_t\}$-stopping time τ.
- $\mathbb{M}$ has the *càdlàg property* if for each $\omega \in \Omega$, the sample path $t \mapsto x_t(\omega)$ is right continuous on $[0, \infty)$ and has left limits on $(0, \infty)$ (inside X_Δ).
- Let (P_t) denote the operator semigroup associated to $\mathbb{M}$ which maps $\mathcal{B}^b(X)$ (the set of all bounded measurable functions on X) into itself given by

$$P_t f(x) = E_x f(x_t),$$

  where E_x is the expectation with respect to P_x. Then a function f is *p-excessive* if it is non-negative and $e^{-pt}P_t f \leq f$ for all $t \geq 0$ and $e^{-pt}P_t f \nearrow f$ as $t \searrow 0$.

4 General Stochastic Hybrid Systems

General Stochastic Hybrid Systems (GSHS) are a class of non-linear stochastic continuous-time hybrid dynamical systems. GSHS are characterized by a hybrid state defined by two components: the continuous state and the discrete state. The continuous and the discrete parts of the state variable have their own natural dynamics, but the main point is to capture the interaction between them.

The time t is measured continuously. The state of the system is represented by a continuous variable x and a discrete variable i. The continuous variable evolves in some "cells"

X^i (open sets in the Euclidean space) and the discrete variable belongs to a countable set Q. The intrinsic difference between the discrete and continuous variables, consists of the way that they evolve through time. The continuous state evolves according to an SDE whose vector field and drift factor depend on the hybrid state. The discrete dynamics produces transitions in both (continuous and discrete) state variables x, i. Switching between two discrete states is governed by a probability law or occurs when the continuous state hits the boundary of its state space. Whenever a switching occurs, the hybrid state is reset instantly to a new state according to a probability law which depends itself on the past hybrid state. Transitions, which occur when the continuous state hits the boundary of the state space are called forced transitions, and those which occur probabilistically according to a state dependent rate are called spontaneous transitions. Thus, a sample trajectory has the form $(q_t, x_t, t \geq 0)$, where $(x_t, t \geq 0)$ is piecewise continuous and $q_t \in Q$ is piecewise constant. Let $(0 \leq T_1 < T_2 < ... < T_i < T_{i+1} < ...)$ be the sequence of jump times.

It is easy to show that GSHS include, as special cases, many classes of stochastic hybrid processes found in the literature PDMP, SHS, etc.

If X is a Hausdorff topological space we use to denote by $\mathcal{B}(X)$ or $\mathcal{B}$ its Borel σ-algebra (the σ-algebra generated by all open sets). A topological space, which is homeomorphic to a Borel subset of a complete separable metric space is called Borel space. A topological space, which is is a homeomorphic with a Borel subset of a compact metric space is called Lusin space.

State space. Let Q be a countable set of discrete states, and let $d : Q \to \mathbb{N}$ and $\mathcal{X} : Q \to \mathbb{R}^{d(.)}$ be two maps assigning to each discrete state $i \in Q$ an open subset X^i of $\mathbb{R}^{d(i)}$. We call the set

$$X(Q, d, \mathcal{X}) = \bigcup_{i \in Q} \{i\} \times X^i$$

the hybrid state space of the GSHS and $x = (i, x^i) \in X(Q, d, \mathcal{X})$ the hybrid state. The closure of the hybrid state space will be

$$\overline{X} = X \cup \partial X$$

where

$$\partial X = \bigcup_{i \in Q} \{i\} \times \partial X^i.$$

It is clear that, for each $i \in Q$, the state space X^i is a Borel space. It is possible to define a metric ρ on X such that $\rho(x_n, x) \to 0$ as $n \to \infty$ with $x_n = (i_n, x_n^{i_n})$, $x = (i, x^i)$ if and only if there exists m such that $i_n = i$ for all $n \geq m$ and $x_{m+k}^i \to x^i$ as $k \to \infty$. The metric ρ restricted to any component X^i is equivalent to the usual Euclidean metric [Dav93]. Each $\{i\} \times X^i$, being a Borel space, will be homeomorphic to a measurable subset of the Hilbert cube, $\mathcal{H}$ (Urysohn's theorem, Prop. 7.2 [BS96]). Recall that $\mathcal{H}$ is the product of countable many copies of $[0, 1]$. The definition of X shows that X is, as well, homeomorphic to a measurable subset of $\mathcal{H}$. Then $(X, \mathcal{B}(X))$ is a Borel space. Moreover, X is a Lusin space because it is a locally compact Hausdorff space with countable base (see [Dav93] and the references therein).

Continuous and discrete dynamics. In each mode X^i, the continuous evolution is driven by the following stochastic differential equation (SDE)

$$dx(t) = b(i, x(t))dt + \sigma(i, x(t))dW_t, \tag{1}$$

where $(W_t, t \geq 0)$ is the m-dimensional standard Wiener process in a complete probability space.

Assumption 1 (Continuous evolution) *Suppose that* $b : Q \times X^{(\cdot)} \to \mathbb{R}^{d(\cdot)}$, $\sigma : Q \times X^{(\cdot)} \to \mathbb{R}^{d(\cdot)\times m}$, $m \in \mathbb{N}$, *are bounded and Lipschitz continuous in* x.

This assumption ensures, for any $i \in Q$, the existence and uniqueness (Theorem 6.2.2. in [Arn74]) of the solution for the above SDE.

In this way, when i runs in Q, the equation (1) defines a family of diffusion processes $\mathbb{M}^i = (\Omega^i, \mathcal{F}^i, \mathcal{F}^i_t, x^i_t, \theta^i_t, P^i)$, $i \in Q$ with the state spaces $\mathbb{R}^{d(i)}$, $i \in Q$. For each $i \in Q$, the elements $\mathcal{F}^i$, $\mathcal{F}^i_t$, θ^i_t, P^i, $P^i_{x^i}$ have the usual meaning as in the Markov process theory (see Appendix).

The jump (switching) mechanism between the diffusions is governed by two functions: the jump rate λ and the transition measure R. The jump rate $\lambda : X \to \mathbb{R}_+$ is a measurable bounded function and the transition measure R maps X into the set $\mathcal{P}(X)$ of probability measures on $(X, \mathcal{B}(X))$. Alternatively, one can consider the transition measure $R : \overline{X} \times \mathcal{B}(X) \to [0,1]$ as a reset probability kernel.

Assumption 2 (Discrete transitions) *(i) for all* $A \in \mathcal{B}(X)$, $R(\cdot, A)$ *is measurable;*
(ii) for all $x \in \overline{X}$ *the function* $R(x, \cdot)$ *is a probability measure.*
(iii) $\lambda : X \to \mathbb{R}_+$ *is a measurable function such that* $t \to \lambda(x^i_t(\omega^i))$ *is integrable on* $[0, \varepsilon(\omega^i))$, *for some* $\varepsilon(\omega^i) > 0$, *for each* $\omega^i \in \Omega^i$.

Since $\overline{X}$ is a Borel space, then $\overline{X}$ is homeomorphic to a subset of the Hilbert cube, $\mathcal{H}$. Therefore, its space of probabilities is homeomorphic to the space of probabilities of the corresponding subset of $\mathcal{H}$ (Lemma 7.10 [BS96]). There exists a measurable function $\digamma : \mathcal{H} \times \overline{X} \to X$ such that $R(x, A) = \mathfrak{p}\digamma^{-1}(A)$, $A \in \mathcal{B}(X)$, where $\mathfrak{p}$ is the probability measure on $\mathcal{H}$ associated to $R(x, \cdot)$ and $\digamma^{-1}(A) = \{\boldsymbol{\omega} \in \mathcal{H} | \digamma(\boldsymbol{\omega}, x) \in A\}$. The measurability of such a function is guaranteed by the measurability properties of the transition measure R.

Construction. We construct an GSHS as a *Markov 'sequence'* H, which admits $(\mathbb{M}^i)$ as subprocesses. The sample path of the stochastic process $(x_t)_{t>0}$ with values in X, starting from a fixed initial point $x_0 = (i_0, x^{i_0}_0) \in X$ is defined in a similar manner as PDMP [Dav93].

Let ω^i be a trajectory which starts in (i, x^i). Let $t_*(\omega^i)$ be the first hitting time of ∂X^i of the process (x^i_t). Let us define the following right continuous multiplicative functional

$$F(t, \omega^i) = I_{(t<t_*(\omega^i))} \exp[-\int_0^t \lambda(i, x^i_s(\omega^i))ds]. \tag{2}$$

This function will be the survivor function for the stopping time S^i associated to the diffusion (x^i_t), which will be employed in the construction of our model. This means that "killing" of the process (x^i_t) is done according to the multiplicative functional $F(t, \cdot)$. The stopping time S^i can be thought of as the minimum of two other stopping times:

1. first hitting time of boundary, i.e. $t_*|_{\Omega^i}$;
2. the stopping time $S^{i\prime}$ given by the following continuous multiplicative functional (which plays the role of the survivor function)

$$M(t, \omega^i) = \exp(-\int_0^t \lambda(i, x^i_s(\omega^i)))ds.$$

The stopping time $S^{i\prime}$ can be defined as

$$S^{i\prime}(\omega^i) = \sup\{t | \Lambda_t^i(\omega^i) \leq m^i(\omega^i)\},$$

where Λ_t^i is the following additive functional associated to the diffusion (x_t^i)

$$\Lambda_t^i(\omega^i) = \int_0^t \lambda(i, x_s^i(\omega^i)))ds$$

and m^i is an $\mathbb{R}_+$-valued random variable on Ω^i, which is exponentially distributed with the survivor function $P_{x^i}^i[m^i > t] = e^{-t}$. Then

$$P_{x^i}^i[S^{i\prime} > t] = P_{x^i}^i[\Lambda_t^i \leq m^i]. \tag{3}$$

We set $\omega = \omega^{i_0}$ and the first jump time of the process is $T_1(\omega) = T_1(\omega^{i_0}) = S^{i_0}(\omega^{i_0})$. The sample path $x_t(\omega)$ up to the first jump time is now defined as follows:

$$\begin{array}{ll} \text{if } T_1(\omega) = \infty: & x_t(\omega) = (i_0, x_t^{i_0}(\omega^{i_0})),\ t \geq 0 \\ \text{if } T_1(\omega) < \infty: & x_t(\omega) = (i_0, x_t^{i_0}(\omega^{i_0})),\ 0 \leq t < T_1(\omega) \\ & x_{T_1}(\omega) \text{ is a r.v. w.r.t. } R((i_0, x_{T_1}^{i_0}(\omega^{i_0})), \cdot). \end{array}$$

The process restarts from $x_{T_1}(\omega) = (i_1, x_1^{i_1})$ according to the same recipe, using now the process $x_t^{i_1}$. Thus if $T_1(\omega) < \infty$ we define $\omega = (\omega^{i_0}, \omega^{i_1})$ and the next jump time

$$T_2(\omega) = T_2(\omega^{i_0}, \omega^{i_1}) = T_1(\omega^{i_0}) + S^{i_1}(\omega^{i_1})$$

The sample path $x_t(\omega)$ between the two jump times is now defined as follows:

$$\begin{array}{ll} \text{if } T_2(\omega) = \infty: & x_t(\omega) = (i_1, x_{t-T_1}^{i_1}(\omega)),\ t \geq T_1(\omega) \\ \text{if } T_2(\omega) < \infty: & x_t(\omega) = (i_1, x_t^{i_1}(\omega)),\ 0 \leq T_1(\omega) \leq t < T_2(\omega) \\ & x_{T_2}(\omega) \text{ is a r.v. w.r.t. } R((i_1, x_{T_2}^{i_1}(\omega)), \cdot). \end{array}$$

and so on.
We denote

$$N_t(\omega) = \sum I_{(t \geq T_k)}$$

Assumption 3 (Non-Zeno executions) *For every starting point $x \in X$, $EN_t < \infty$, for all $t \in \mathbb{R}_+$.*

We can now define GSHS by:

Definition 1 *A General Stochastic Hybrid System (GSHS) is a collection $H = ((Q, d, \mathcal{X}), b, \sigma, Init, \lambda, R)$ where*

- *Q is a countable set of discrete variables;*
- *$d : Q \to \mathbb{N}$ is a map giving the dimensions of the continuous state spaces;*
- *$\mathcal{X} : Q \to \mathbb{R}^{d(\cdot)}$ maps each $q \in Q$ into an open subset X^q of $\mathbb{R}^{d(q)}$;*
- *$b : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(\cdot)}$ is a vector field;*
- *$\sigma : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(\cdot) \times m}$ is a $X^{(\cdot)}$-valued matrix, $m \in \mathbb{N}$;*

- $Init : \mathcal{B}(X) \to [0,1]$ *is an initial probability measure on* $(X, \mathcal{B}(S))$*;*
- $\lambda : \overline{X}(Q, d, \mathcal{X}) \to \mathbb{R}^+$ *is a transition rate function;*
- $R : \overline{X} \times \mathcal{B}(\overline{X}) \to [0,1]$ *is a transition measure.*

Following [Sie81], we note that if R_c is a transition measure from $(X \times Q, \mathcal{B}(X \times Q))$ to $(X, \mathcal{B}(X))$ and R_d is a transition measure from $(X, \mathcal{B}(X))$ to $(Q, \mathcal{B}(Q))$ (where Q is equipped with the discrete topology) then one might define a transition measure as follows

$$R(x^i, A) = \sum_{q \in Q} R_d(x^i, q) R_c(x^i, q, A^q)$$

for all $A \in \mathcal{B}(X)$, where $A^q = A \cap (q, X^q)$. Taking in the definition of a GSHS a such kind of reset map, the change of the continuous state at a jump depends on the pre jump location (continuous and discrete) as well as on the post jump discrete state.
This construction can be used to prove that the stochastic hybrid processes with jumps, developed in [Blo03], are a particular class of GSHS.

Also we can define GSHS executions as:

Definition 2 (GSHS Execution) *A stochastic process* $x_t = (q(t), x(t))$ *is called a GSHS execution if there exists a sequence of stopping times* $T_0 = 0 < T_1 < T_2 \leq \ldots$ *such that for each* $k \in \mathbb{N}$,

- $x_0 = (q_0, x_0^{q_0})$ *is a* $Q \times X$*-valued random variable extracted according to the probability measure* $Init$*;*
- *For* $t \in [T_k, T_{k+1})$, $q_t = q_{T_k}$ *is constant and* $x(t)$ *is a (continuous) solution of the SDE:*

$$dx(t) = b(q_{T_k}, x(t))dt + \sigma(q_{T_k}, x(t))dW_t \tag{4}$$

  where W_t *is a the* m*-dimensional standard Wiener;*
- $T_{k+1} = T_k + S^{i_k}$ *where* S^{i_k} *is chosen according with the survivor function (2).*
- *The probability distribution of* $x(T_{k+1})$ *is governed by the law* $R\left((q_{T_k}, x(T_{k+1}^-)), \cdot\right)$.

5 Markov strings

In this section we formulate a very general class of Markov processes, which will be called *Markov strings*, loosely based on the so-called "melange" operation of Markov processes [Mey75]. A Markov string is a hybrid state 'jump Markov process'. The 'continuous state' component switches back and forth at random moments of times among a countable collections of Markov processes defined on some evolution modes. The 'discrete component' keeps track of the index of which Markov process the continuous component is following. This discrete component plays the role of an 'evolution index'. The continuous state is allowed to jump whenever the evolution index changes. For a Markov string the sojourn time in each mode is given as a stopping time with memoryless property for the process which evolves in that mode. Moreover, the continuous state immediately before a switching between modes is allowed to influence that jump.

5.1 Informal description

We start with:

1. a countable family of independent Markov processes with some nice properties, for example the strong Markov property, the càdlàg property.

2. a sequence of independent stopping times (for each process is given a stopping time with memoryless property).

3. a renewal kernel is a priory given.

The stopping times play the role of the jump times from one process to another and the renewal kernel gives the distribution of the post-jump state. The probabilistic construction of the Markov string is natural:
1. start with one process, which belongs to the given family;
2. kill the current process at the corresponding stopping time;
3. jump according to the renewal kernel;
4. restart another process (belonging to the given family) from the new state;
5. return to 2. and repeat.
The pieced together process obtained by the above procedure is called Markov string. The main aim of this section is to prove that the Markov string inherits the properties (like the strong Markov property and the càdlàg property) from its component processes.
The Markov string construction is closely related to the mixing operation of Markov processes from [Mey75] and the random evolution process construction from [Sie81].Markov strings differ from the class of processes considered in [Mey75], in that:
1. The jump times are essentially given stopping times, *not necessarily the life times of the component processes*; 2. After a jump, the string is allowed to restart following another process, which might be different from the pre-jump process.
The mixing ("melange") operation in [Mey75] is only sketched and the author claims that it can be obtained using the renewal ("renaissance") operation. We consider that the passing from renewal to mixing is not straightforward. It is necessary to emphases the construction of all probabilistic elements associated with the resulted string. Lifting the renewal construction to the mixing construction, remarkable changes should be introduced in the Markov string definitions of the state space, probability space, probabilities on the trajectories.

As well, Markov strings can be obtained by specializing the base process and the 'instantaneous' distribution in the structure of the random evolution processes developed by Siegrist in [Sie81], but the proof of the strong Markov property is not given in [Sie81]. There, the author claims this can be derived from the strong Markov property of revival processes introduced by Ikeda, et. al. in [INW66]. To our knowledge, this property is completly proved by Meyer, in [Mey75], for revival processes.

5.2 The Ingredients

Suppose that $\mathbb{M}^i = (\Omega^i, \mathcal{F}^i, \mathcal{F}^i_t, x^i_t, \theta^i_t, P^i, P^i_{x^i})$, $i \in Q$ is a countable family of Markov processes. We denote the state space of each $\mathbb{M}^i$ by $(X^i, \mathcal{B}^i)$ and assume that $\mathcal{B}^i$ is the Borel σ-algebra of X^i if X^i is a topological Hausdorff space. We denote by Δ the cemetery point for all X^i, $i \in Q$. The existence of Δ is assumed for reasons that will be

clear below. For each $i \in Q$, the elements $\mathcal{F}^i$, $\mathcal{F}^{i,0}_t$, $\mathcal{F}^i_t$, θ^i_t, P^i, $P^i_{x^i}$ have the usual meaning as in the Markov process theory.

Let (P_t^i) denote the operator semigroup associated to $\mathbb{M}^i$, which maps $\mathcal{B}^i(X^i)$ into itself, given by

$$P_t^i f^i(x^i) = E_{x^i}^i f^i(x_t^i),$$

where $E_{x^i}^i$ is the expectation w.r.t. $P_{x^i}^i$. Then a function f^i is *p-excessive* $(p > 0)$ w.r.t. $\mathbb{M}^i$ if $f^i \geq 0$ and $e^{-pt}P_t^i f^i \leq f^i$, for all $t \geq 0$ and $e^{-pt}P_t^i f^i \nearrow f^i$ as $t \searrow 0$.

Assumption 4 *For each $i \in Q$, we suppose that:*
1. $\mathbb{M}^i$ is a strong Markov process.
2. P^i is a complete probability.
3. The state space X^i is a Borel space.
4. $\mathbb{M}^i$ enjoys the càdlàg property, i.e. for each $\omega^i \in \Omega^i$, the sample path $t \mapsto x_t^i(\omega^i)$ is right continuous on $[0, \infty)$ and has left limits on $(0, \infty)$ (inside X_Δ^i).
5. The p-excessive functions of $\mathbb{M}^i$ are P^i-a.s. right continuous on trajectories.

Part 3. implies that the underlying probability space Ω^i can be assumed to be $D_{[0,\infty)}(X^i)$, the space of functions mapping $[0, \infty)$ to X^i which are right continuous functions with left limits. Let us consider ω_Δ^i the cemetery point of Ω^i corresponding to the 'dead' trajectory of $\mathbb{M}^i$ (when the process is trapped to Δ).

In the terminology of [Mey66], parts 1., 3. and 5. of the Assumption 4 imply that each $\mathbb{M}^i$ is a *right process.*

Using this family of Markov processes $\{\mathbb{M}^i\}_{i \in Q}$, we define a new Markov process whose realizations consist of concatenations of realizations for different $\mathbb{M}^i$. To achieve this goal, we need to define the transition mechanism from one process to the others. The jumping mechanism will be driven by:
1. A stopping time (which gives the jump temporal parameter) for each process;
2. A renewal kernel, which gives the post jump state.

Formally, in order to define the desired Markov string, $\mathbb{M}$, we need to give:
1. $(S^i)_{i \in Q}$, where, for each $i \in Q$, S^i is a *stopping time* of $\mathbb{M}^i$,
2. The jumping mechanism between the processes $\mathbb{M}^i$ is governed by a *renewal kernel,* which is a Markovian kernel

$$\Psi : \{\bigcup_{i \in Q} \Omega^i\} \times \mathcal{B}(X) \to [0, 1]$$

Assumption 5 *(i) For each $i \in Q$, S^i is terminal time, i.e. stopping time with the 'memoryless' property:*

$$S^i(\theta_t^i \omega^i) = S^i(\omega^i) - t, \, \forall t < S^i(\omega^i) \tag{5}$$

(ii) The renewal kernel Ψ satisfies the following conditions: (a) If $S^i(\omega^i) = +\infty$ then $\Psi(\omega^i, \cdot) = \varepsilon_\Delta$ (here, ε_Δ is the Dirac measure corresponding to Δ); (b) If $t < S^i(\omega^i)$ then $\Psi(\theta_t^i \omega^i, \cdot) = \Psi(\omega^i, \cdot)$.

Note that the component processes have the càdlàg property, therefore they may also have jumps, which are not treated separately in the construction of the Markov strings. The sequence of jump times refers to additional jumps, not to the jumps of the trajectories of component processes.

We consider now, for each $i \in Q$, the killed process $\widetilde{\mathbb{M}}^i = (\Omega^i, \mathcal{F}^i, \mathcal{F}_t^i, \widetilde{x}_t^i, \widetilde{\theta}_t^i, P^i, P_{x^i}^i)$ where

$$\widetilde{x}_t^i(\omega^i) = \begin{cases} x_t^i(\omega^i), & \text{if } t < S^i(\omega^i) \\ \Delta, & \text{if } t \geq S^i(\omega^i) \end{cases} \quad \text{and} \quad \widetilde{\theta}_t^i(\omega^i) = \begin{cases} \theta_t^i(\omega^i), & \text{if } t < S^i(\omega^i) \\ \omega_\Delta^i, & \text{if } t \geq S^i(\omega^i) \end{cases}$$

In this case, Ω^i should be thought of as a subspace of $\Omega^i \times [0,\infty)$, the above embedding is made through the map $\omega^i \mapsto (\omega^i, S^i(\omega^i))$. The killed process is equivalent with the subprocess of $\mathbb{M}^i$ corresponding to the multiplicative functional $M_t^i = I_{[0,S^i)}(t)$ (see Chapter III, [BG68]).

5.3 The Construction

Using the elements defined in the section 5.2 we construct the pieced-together stochastic process $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$, which will be called *Markov string*. We have to point out that $\mathbb{M}$ is obtained by the concatenation of the killed processes $\widetilde{\mathbb{M}}^i$.

To completely define the Markov string we need to specify the following elements: 1. $(X, \mathcal{B})$ - the state space; 2. $(\Omega, \mathcal{F}, P)$ - the underlying probability space; 3. $\mathcal{F}_t$ - the natural filtration; 4. θ_t - the translation operator; 5. P_x - Wiener probabilities.

State Space $(X, \mathcal{B})$. The state space will be X defined as follows. X is constructed as the direct sum of spaces X^i, with the same cemetery point Δ, i.e.

$$X = \bigcup_{i\in Q} \{(i,x) | x \in X^i\}. \tag{6}$$

In the same manner as in the section 4, it results that X is a Borel space.

The space X can be endowed with the Borel σ-algebra $\mathcal{B}(X)$ generated by its metric topology. Moreover, we have

$$\mathcal{B}(X) = \sigma\{\bigcup_{i\in Q} \{i\} \times \mathcal{B}^i\}. \tag{7}$$

Then $(X, \mathcal{B}(X))$ is a Borel space, whose Borel σ-algebra $\mathcal{B}(X)$ restricted to each component X^i gives the initial σ-algebra $\mathcal{B}^i$ [Dav93].

We can assume, without loss of generality, that $X^i \cap X^j = \emptyset$ if $i \neq j$. Thus the relations (6) and (7) become

$$X = \bigcup_{i\in Q} X^i; \tag{8}$$

$$\mathcal{B}(X) = \sigma(\bigcup_{i\in Q} \mathcal{B}^i). \tag{9}$$

Therefore, we can assume, as well, that $\Omega^i \cap \Omega^j = \emptyset$ if $i \neq j$.

Probability Space. The space Ω can be thought as the space generated by the concatenation operation defined on the union of the spaces Ω^i (which are pairwise disjoint), i.e. $\Omega = (\bigcup_{i\in Q} \Omega^i)^*$. Note that, for each $i \in Q$, an arbitrary element ω^i of Ω^i must be thought as a trajectory of the killed process $\widetilde{\mathbb{M}}^i$. The cemetery point of Ω is denoted by $\omega_\Delta = (\omega_\Delta^i)_{i\in Q}$. We use to denote by ω (resp. $\widehat{\omega}$ or ω^i) an arbitrary element of Ω (resp. $\bigcup\limits_{i\in Q} \Omega^i$ or Ω^i).

The $\sigma-$algebra $\mathcal{F}$ on Ω will be the smallest $\sigma-$algebra on Ω such that the projection $\pi^i : \Omega \to \Omega^i$ are $\mathcal{F}/\mathcal{F}^i$ measurable, $i \in Q$. The probability P on $\mathcal{F}$ will be defined as a 'product measure'. Let $\widehat{\mathcal{F}}$ be the $\sigma(\bigcup\limits_{i\in Q} \mathcal{F}^i)$ defined on $\bigcup\limits_{i\in Q} \Omega^i$.

Recipe. We give the procedure to construct a sample path of the stochastic process $(x_t)_{t>0}$ with values in X, starting from a fixed initial point $x_0 = x_0^{i_0} \in X^{i_0}$. Let ω^{i_0} be a sample path of the process $(x_t^{i_0})$ starting with x_0. In fact, we give a recipe *to construct a Markov string* *starting with an initial path* ω^{i_0}. Let $T_1(\omega^{i_0}) = S^{i_0}(\omega^{i_0})$. The event ω and the associated sample path are inductively defined. In the first step

$$\omega = \omega^{i_0}$$

The sample path $x_t(\omega)$ up to the first jump time is now defined as follows:

$$\begin{array}{ll} \text{if } T_1(\omega) = \infty : & x_t(\omega) = x_t^{i_0}(\omega^{i_0}),\ t \geq 0 \\ \text{if } T_1(\omega) < \infty : & x_t(\omega) = x_t^{i_0}(\omega^{i_0}),\ 0 \leq t < T_1(\omega) \\ & x_{T_1} \text{ is a r.v. according to } \Psi(\omega^{i_0}, \cdot). \end{array}$$

The process restarts from $x_{T_1} = x_1^{i_1}$ according to the same recipe, using now the process $(x_t^{i_1})$. Let ω^{i_1} be a sample of the process $(x_t^{i_1})$ starting with $x_1^{i_1}$. Thus, if $T_1(\omega) < \infty$ we define the next jump time

$$T_2(\omega^{i_0}, \omega^{i_1}) = T_1(\omega^{i_0}) + S_{i_2}(\omega^{i_2}).$$

Then, in the second step

$$\omega = \omega^{i_0} * \omega^{i_1}$$

where '$*$' is the concatenation operation of trajectories. The sample path $x_t(\omega)$ between the two jump times is now defined as follows:

$$\begin{array}{ll} \text{if } T_2(\omega) = \infty : & x_t(\omega) = x_{t-T_1}^{i_1}(\omega^{i_1}),\ t \geq T_1(\omega) \\ \text{if } T_2(\omega) < \infty : & x_t(\omega) = x_t^{i_1}(\omega^{i_1}),\ 0 \leq T_1(\omega) \leq t < T_2(\omega) \\ & x_{T_2} \text{ is a r.v. according to } \Psi(\omega^{i_1}, \cdot). \end{array}$$

Generally, if $T_k(\omega) = T_k(\omega^{i_0}, \omega^{i_1}, ..., \omega^{i_{k-1}}) <$ with

$$\omega = \omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{k-1}}$$

then the next jump time is

$$T_{k+1}(\omega) = T_{k+1}(\omega^{i_0}, \omega^{i_1}, ..., \omega^{i_k}) = T_k(\omega^{i_0}, \omega^{i_1}, ..., \omega^{i_{k-1}}) + S^{i_k}(\omega^{i_k}) \tag{10}$$

The sample path $x_t(\omega)$ between the two jump times T_k and T_{k+1} is defined as:

$$\begin{array}{lcll} \text{if } T_{k+1}(\omega) & = & \infty : & x_t(\omega) = x_{t-T_k}^{i_k}(\omega^{i_k}), t \geq T_{k+1}(\omega) \\ \text{if } T_{k+1}(\omega) & < & \infty : & \begin{array}{l} x_t(\omega) = x_{t-T_k}^{i_k}(\omega^{i_k}), 0 \leq T_k(\omega) \leq t < T_{k+1}(\omega) \\ x_{T_{k+1}} \text{ is a r.v. according to } \Psi(\omega^{i_k}, \cdot). \end{array} \end{array} \tag{11}$$

We have constructed a sequence of jump times $0 < T_1 < T_2 < ... < T_n < ...$ Let $T_\infty = \lim\limits_{n\to\infty} T_n$. Then $x_t(\omega) = \Delta$ if $t \geq T_\infty$. A sample path until T_{k_0} (where $k_0 = \min\{k : S^{i_k}(\omega) = \infty\}$) of the process (x_t), starting from a fixed initial point $x_0 = (i_0, x_0^{i_0})$, is obtained as the concatenation:

$$\omega = \omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{k_0-1}}.$$

We denote $N_t(\omega) = \sum I_{(t \geq T_k)}$ the number of jump times in the interval $[0, t]$. To eliminate pathological solutions that take an infinite number of discrete transitions in a finite amount of time (known as Zeno solutions) we impose the following assumption:

Assumption 6 (Non-zeno dynamics) *For every starting point* $x \in X$, $EN_t < \infty$, *for all* $t \in \mathbb{R}_+$.

Under Assumption 6, the underlying probability space Ω can be identified with $D_{[0,\infty)}(X)$.

Wiener Probabilities. One might define the expectation $E^x f$, $x \in X$, where f is a $\mathcal{F}$-measurable function on Ω, which depends only on a finite number of variables, by recursion on the number of variables.
Step1. If $\omega = \omega^{i_0}$and $f(\omega) = f_1(\omega^{i_0})$ with f_1 a $\mathcal{F}^{i_0}$-measurable function on Ω^{i_0}, then
• if $x = x^{i_0} \in X^{i_0}$ then $E_x f = E^{i_0}_{x^{i_0}} f$, where $E^{i_0}_{x^{i_0}}$ is the expectation corresponding to the probability $P^{i_0}_{x^{i_0}}$;
• if $x = x^j \in X^j$, $j \neq i_0$ then $E_x f = 0$.
Step2. If $\omega = \omega^{i_0} * \omega^{i_1} * ... * \omega^{i_n}$and $f(\omega) = f_n(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_n})$ with f_n a $\prod\limits_{k=0}^{n} \mathcal{F}^{i_k}$-measurable function on $\prod\limits_{k=0}^{n} \Omega^{i_k}$ then

$$\begin{aligned} f_{n-1}(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{n-1}}) &= \int_{\Omega^{i_n}} f_n(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{n-1}} * \omega^{i_n}) dP^{i_n}_{\Psi(\omega^{i_{n-1}},\cdot)}(\omega^{i_n}); \\ g(\omega) &= f_{n-1}(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{n-1}}); \\ E_x f &= E_x g. \end{aligned} \tag{12}$$

Translation Operators. Let us define now the translation operator (θ_t) associated with (x_t). If $t \geq T_\infty(\omega)$, then we take $\theta_t(\omega) = \omega_\Delta$. Otherwise, there exists k such that $T_k(\omega) \leq t < T_{k+1}(\omega)$. In this case we take

$$\theta_t(\omega) = (\theta^{i_k}_{t-T_k(\omega)}(\omega^{i_k}) * \omega^{i_{k+1}} * ...). \tag{13}$$

Lemma 1 *(θ_t) is the translation operator associated with (x_t), i.e.*

$$\theta_s \circ \theta_t = \theta_{s+t}; \;\; x_s \circ \theta_t = x_{s+t}.$$

Proof. If $t \geq T_\infty(\omega)$, then $\theta_t(\omega) = \omega_\Delta$ and $x_{s+t}(\omega) = \Delta = x_s(\theta_t(\omega))$.
Suppose that there exist $k, i \geq 0$ such that $T_k(\omega) \leq t < T_{k+1}(\omega)$ and $T_i(\theta_t\omega) \leq s < T_{i+1}(\theta_t\omega)$. Then

$$x_t(\omega) = x^{i_k}_{t-T_k}(\omega^{i_k}); \;\; (x_s \circ \theta_t)(\omega) = x^{i_l}_{s-T_l}(\theta^{i_l}_{s-T_l}\omega^{i_l}).$$

Since $\theta_t(\omega)$ is given by (13) and T_{k+1} is given by (10) we obtain

$$\begin{aligned} T_{k+1}(\theta_t\omega) &= S^{i_k}(\theta^{i_k}_{t-T_k(\omega)}(\omega^{i_k})) = S^{i_k}(\omega^{i_k}) - (t - T_k(\omega)) \\ &= T_{k+1}(\omega) - t. \end{aligned}$$

Then

$$T_{i+1}(\theta_t\omega) = T_{k+i+1}(\omega) - t$$

Therefore

$$T_i(\theta_t\omega) \leq s < T_{i+1}(\theta_t\omega) \Leftrightarrow T_{k+i}(\omega) \leq s + t < T_{k+i+1}(\omega).$$

■

Natural Filtrations. Let $(\mathcal{F}_t)$ be the natural filtration with respect to (x_t). The natural filtration $(\mathcal{F}_t)$ on Ω is built such that we have the following definition of $\mathcal{F}_t$-measurability:

Definition 3 *A $\mathcal{F}$-measurable function f on Ω is $\mathcal{F}_t$-measurable if the following property holds:*
For each k, the function $f \cdot I_{\{T_k(\omega) \le t < T_{k+1}(\omega)\}}$ is equal to $h \circ \eta_k$, where the function $h(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k})$ is such that for a fixed $(\widehat{\omega}^{i_0} * \widehat{\omega}^{i_2} * ... * \widehat{\omega}^{i_{k-1}})$ with $T_k(\widehat{\omega}^{i_0} * \widehat{\omega}^{i_2} * ... * \widehat{\omega}^{i_{k-1}}) \le t$, $\omega^{i_k} \mapsto h(\widehat{\omega}^{i_0} * \widehat{\omega}^{i_2} * ... * \widehat{\omega}^{i_{k-1}} * \omega^{i_k})$ is measurable with respect to $\mathcal{F}^{i_k}_{t-T_k}$.

Because the families of filtrations $(\mathcal{F}^i_t)$ are nondecreasing and right continuous, one can verify that the family $(\mathcal{F}_t)$ has the same properties, as follows.

Proposition 2 *(i) The family $(\mathcal{F}_t)$ is nondecreasing and right continuous.*
(ii) The random variables T_k are stopping times w.r.t. $(\mathcal{F}_t)$.
(iii) Let T a stopping time with respect to $(\mathcal{F}_t)$. For each $k \in \mathbb{N}$, $T \wedge T_k$ is a function on Ω which depends only on $\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{k-1}}$. On the other hand, if $\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{k-1}}$ is fixed, the function $(T \wedge T_{k+1} - T_k)^+$ with ω^{i_k} as argument is a stopping time with respect $(\mathcal{F}^{i_k}_t)$.

Proof. The proof can be obtained with small changes from the similar result proofs given in [Mey75] for the case of rebirth processes. ∎

5.4 Basic Properties

Mainly, in this section we prove that the Markov string (x_t) constructed in section 5.3 is a right Markov process. The proof engine is based on the Markov property of the discrete time Markov chain (p_n), which will be build in the following.

(p_n) is a discrete time Markov chain associated to (x_t) with the state space $(\bigcup_{i \in Q} \Omega_i, \widehat{\mathcal{F}})$ and the underlying probability space $(\Omega, \mathcal{F})$. The chain (p_n) is essentially 'the $n-th$' step of the process (x_t). If its starting point is ω^{i_0} (a trajectory in Ω^{i_0} starting in $x^{i_0}_0$) then $p_n(\omega) = \omega^{i_n}$.
The transition kernel associated with (p_n) can be defined as follows: $H(\widehat{\omega}, A) = P_\Psi(\widehat{\omega}, A)$, $A \in \widehat{\mathcal{F}}$. The construction of P_x from subsection 5.3 is such that

- H is the transition function of (p_n);
- P_x is the initial probability law of (p_n); i.e. if $\widehat{\omega} \in \bigcup_{i \in Q} \Omega_i$ which starts in $x \in X$

$$P^{\widehat{\omega}}(p_0 \in A) = P_x(A), \; A \in \mathcal{F}.$$

Let η_k be the projection $(p_0, p_1, ..., p_k)$, i.e. $\eta_k(\omega) = (\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k})$.

One might construct a jump process (η_t) associated to a Markov string (x_t) following a similar algorithm such that used for Piecewise Deterministic Markov processes, in [Dav93]. We do not have a one-to-one correspondence between the sample paths of (x_t) and (η_t), as in the case of PDMP. Then the jump process will not serve to study the Markov string. Its role is taken by the Markov chain (p_n).

Remark 1 *For each k on the set $\{T_k(\omega) \le t < T_{k+1}(\omega)\}$ we have: $x_t = x^{i_k}_{t-T_k} \circ p_k$.*

Proposition 3 (Simple Markov property) *Under Assumptions 4-6, any Markov string $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$ is a Markov process.*

Proof. The simple Markov property of (x_t) is equivalent to the following implication [Mey75]:

If f is a positive $\mathcal{F}_t$-measurable function and g is a $\mathcal{F}$-measurable function then

$$E^x[f \cdot g \circ \theta_t] = E^x[f \cdot E^{x_t}[g]]. \tag{14}$$

The identity (14) can be unfolded into two separated equalities

$$\begin{aligned} E^x[f \cdot g \circ \theta_t \cdot I_{\{t \geq T_\infty\}}] &= E^x[f \cdot E^{x_t}[g] \cdot I_{\{t \geq T_\infty\}}] \quad (15)\\ E^x[f \cdot g \circ \theta_t \cdot I_{\{T_k(\omega) \leq t < T_{k+1}(\omega)\}}] &= E^x[f \cdot E^{x_t}[g] \cdot I_{\{T_k(\omega) \leq t < T_{k+1}(\omega)\}}] \quad (16)\end{aligned}$$

The identity (15) is clear because on $\{t \geq T_\infty\}$

$$E^{x_t}[g] = g(\omega_\Delta);\ \theta_t(\omega) = \omega_\Delta;\ x_t(\omega) = \Delta.$$

Let us prove now the identity (16). Let $\omega \in \Omega$. By the definition of $\mathcal{F}_t$ we have

$$f(\omega) \cdot I_{\{T_k(\omega) \leq t < T_{k+1}(\omega)\}}(\omega) = h(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k}) \tag{17}$$

where h is a measurable function as in the definition 3 and is equal to zero outside of the set $\{T_k(\omega) \leq t < T_{k+1}(\omega)\}$.

In order to prove (16) it is enough to treat the case when the function g depends only on a finite number of variables (because the expectation E^x is defined by the recursion (12)).

We start with the case when the function g depends only on a single variable, ω^{i_0}, i.e. $g(\omega) = a(\omega^{i_0})$, where a is $\mathcal{F}^{i_0}$-measurable on Ω^{i_0}. In this case, the left-hand side of (16) is equal to

$$E^x[f \cdot I_{\{T_k(\omega) \leq t < T_{k+1}(\omega)\}} \cdot a(\theta^{i_k}_{t-T_k(\omega)}(\omega^{i_k}))]. \tag{18}$$

Because the term between [...] depends only on $(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k})$, (18) becomes

$$E^x\{\int_{\Omega^{i_k}} h(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k}) \cdot a(\theta^{i_k}_{t-T_k(\omega)}(\omega^{i_k}))dP^{i_k}_{\Psi(\omega^{i_{k-1}},\cdot)}(\omega^{i_k})\}. \tag{19}$$

Again, the integrand between $\{...\}$ depends only on $(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{k-1}})$. Since the function $\omega^{i_k} \to h(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k})$ is $\mathcal{F}^{i_k}_{t-T_k}$-measurable, we can use the Markov property of the process $\mathbb{M}^{i_k}$ and (19) becomes

$$\int_{\Omega^{i_k}} h(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k}) E^{i_k}_{x^{i_k}_{t-T_k}(\omega^{i_k})}[a] dP^{i_k}_{\Psi(\omega^{i_{k-1}},\cdot)}(\omega^{i_k}). \tag{20}$$

Since $x_t(\omega) = x^{i_k}_{t-T_k}(\omega^{i_k})$ on $\{T_k(\omega) \leq t < T_{k+1}(\omega)\}$ the computation of the right-hand side of (16) gives

$$E^x\{h(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k}) \cdot E^{i_k}_{x^{i_k}_{t-T_k}(\omega^{i_k})}[a]\} \tag{21}$$

Using the recursive procedure, as before, (21) gives (20).

Suppose now that (16) is established for all functions g which depend only on $(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{k-1}})$. We have to prove that (16) is true for

$$g(\omega) = g(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k});\ k > 0.$$

Let

$$c(\omega) = c(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_{k-1}}) = \int_{\Omega^{i_k}} b(\omega^{i_0} * \omega^{i_1} * ... * \omega^{i_k}) dP^{i_k}_{\Psi(\omega^{i_{k-1}},\cdot)}(\omega^{i_k}).$$

Using the recursive procedure, one can check that the functions

$$h(...)g \circ \theta_t \quad \text{and} \quad h(...)c \circ \theta_t$$

have the same expectations.
On the other hand, the functions

$$h(...)E_{x_t}[g] \quad \text{and} \quad h(...)E_{x_t}c$$

have the same expectations. Since c depends only on $k-1$ variables, this implies (16) for the general case. ■

Proposition 4 (Cadlag property) *Under Assumptions 4-6, any Markov string* $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$ *has the cadlag property, i.e. for all* $\omega \in \Omega$ *the trajectories* $t \mapsto x_t(\omega)$ *are right continuous on* $[0, \infty)$ *with left limits on* $(0, \infty)$.

Proof. The result is a direct consequence of two facts:
1. the sample paths of (x_t) are obtained by the concatenation of sample paths of component process (i.e. the concatenation is done in such way it preserves the right continuity and the left limits);
2. the component processes enjoy the càdlàg property.
Then the Markov string inherits the càdlàg property. ■

Proposition 5 *Under Assumptions 4-6, any Markov string* $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$ *is a strong Markov process.*

Proof. Each T_k is a stopping time for (x_t) (see proposition 2 (ii)). For each $k \geq 1$, T_k can be obtained by the following recursion

$$T_{k+1} = T_k + S^{i_k} \circ \theta_{T_k}$$

Let us prove now that the process (x_t) is a strong Markov process. The filtration $(\mathcal{F}_t)$ is nondecreasing and right continuous (see proposition 2 (i)). Then the process (x_t) satisfies the right hypothesis.
Let (P_t) be the semigroup of the whole Markov process (x_t), $P_t g(x) = E_x g(x_t)$, where g is bounded $\mathcal{B}$-measurable function. Let $(U_p)_{p>0}$ the resolvent associated to the semigroup, i.e.

$$U_p g = \int_0^\infty e^{-pt} P_t g dt.$$

It is known that the strong Markov property is equivalent with each from the following assertions [Mey67]:
1. If g is a positive bounded continuous function on X_Δ then $f = U_p g$ ($p > 0$) is nearly Borel and right continuous on the process trajectories.
2. Each p-excessive function ($p > 0$) is nearly Borel and right continuous on the process trajectories.
Recall that a real function defined on the state space X_Δ is nearly Borel for the process (x_t) if there exist two Borel function h and h' on X_Δ such that $h' \leq f \leq h$ and

$$P\{\omega | \exists t, h' \circ x_t(\omega) < h \circ x_t(\omega)\} = 0. \tag{22}$$

Let g be a positive bounded continuous function on X. We have $g = \sum_{i \in Q} g^i$, where $g^i = g|_{X^i}$ are bounded continuous functions on X^i. Then $P_t g = \sum_{i \in Q} P^i_t g^i$ and

$$U_p g = \int_0^\infty e^{-pt} P_t g dt = \sum_{i \in Q} \int_0^\infty e^{-pt} P^i_t g^i dt = \sum_{i \in Q} U^i_p g^i.$$

It is known that $f = U_p g$ $(p > 0)$ (the restriction to X) is p-excessive function with respect to (P_t) and for each $i \in Q$ and the function $f^i = U^i_p g^i$ is p-excessive function with respect to (P^i_t). Therefore, f^i is nearly Borel and right continuous on the trajectories of the process (x^i_t). It is clear from the construction that the function f is right continuous on the trajectories of the process (x_t).
Let $h^i, h^{i\prime}$ two Borel functions on X^i_Δ such that $h' \leq f^i \leq h^i$ and

$$h^{i\prime} \circ x^i_t(\omega^i) = h^i \circ x^i_t(\omega^i) \ P^i - a.s., \forall t \geq 0. \tag{23}$$

Let us consider the function h, h' defined as below:

$$h = \sum_{i \in Q} h^i, \ h' = \sum_{i \in Q} h^{i\prime} \ . \tag{24}$$

It is clear that

$$P\{\omega | \exists t \geq T_\infty, h' \circ x_t(\omega) < h \circ x_t(\omega)\} = 0.$$

Let us compute the probability of the following event:

$$A_k = \{\exists t | T_k \leq t < T_{k+1}, h' \circ x_t(\omega) < h \circ x_t(\omega)\}.$$

We have $A_k \in \mathcal{F}$. Let $a_k = I_{A_k}$ which depends only on $\omega^{i_0} * \omega^{i_2} * ... * \omega^{i_k}$. The recursive method to compute the probability of A_k on $\{T_k \leq t < T_{k+1}\}$ gives

$$\int_{\Omega^{i_k}} a_k(\omega^{i_0} * \omega^{i_2} * ... * \omega^{i_k}) dP^{i_k}_{\Psi(\omega^{i_{k-1}}, \cdot)}(\omega^{i_k}). \tag{25}$$

Since $a_k(\omega^{i_0} * \omega^{i_2} * ... * \omega^{i_k})$ on Ω^{i_k} is exactly the indicator function of

$$B = \{\omega^{i_k} | \exists u < S^{i_k}(\omega^{i_k}), h^{i_k\prime} \circ x^{i_k}_u(\omega) < h^{i_k} \circ x^{i_k}_u(\omega)\}$$

using (23) we obtain that the integral (25) is zero. Therefore the functions h, h' defined by (24) verify the condition (22). Then f will be a nearly Borel function relative to the process (x_t). ■

The Propositions 3, 4, 5 can be summarized in the following theorem:

Theorem 6 *Under Assumptions 4-6, any Markov string* $\mathbb{M} = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P, P_x)$ *has the following properties:*
(i) It is a strong Markov process;
(ii) It has the cadlag property;
(iii) It is a right process.

6 Properties of GSHS

Strong Markov property. GSHS, being constructed as particular Markov strings, they inherit the properties of their diffusion component, namely they are *strong Markov processes* with *càdlàg property*.

Proposition 7 (Strong Markov process) *Under the standard assumptions 1-3, any General Stochastic Hybrid Model H is a strong Markov process.*

Proof. To prove that H is a strong Markov process, it is enough to check that a GSHS is, indeed, a Markov string, i.e. it satisfies the Assumptions 4-6 from the Markov string construction. It is easy to see that

- Ass.1 implies Ass.4;
- Ass.3 implies Ass.6.

It remains to prove only that Assumption 2 and the construction of a GSHS implies Assumption 5. We can suppose without loss of generality that $\Omega^i \cap \Omega^j = \emptyset$. Then, the kernel Ψ can be defined as follows

$$\Psi : \{\bigcup_{i \in Q} \Omega^i\} \times \mathcal{B}(X) \to [0,1] \quad \text{such that} \quad \Psi(\omega^i, A) = R(x^i_{S^i(\omega^i)}, A)$$

For any GSHS, we need to check
(a) the memoryless property of kernel, i.e. if $0 < t < S^i(\omega^i)$ then $\Psi(\theta^i_t \omega^i, \cdot) = \Psi(\omega^i, \cdot) \Leftrightarrow R(x^i_{S^i(\theta^i_t \omega^i)}, \cdot) = R(x^i_{S^i(\omega^i)}, \cdot)$.
(b) the memoryless property of the stopping times S^i.
Since the component diffusions are strong Markov processes (b) implies (a). In fact, we have to prove that, if $0 < t < t + s < S^i(\omega^i)$ then stopping times (S^i)

$$P_{x^i}(S^i > t + s | S^i > t) = P_{x^i_t}(S^i > s) \tag{26}$$

We have, for each $i \in Q$,

1. the hitting time of the boundary ∂X^i of the diffusion process (x^i_t) has the memoryless property, i.e. $t^*(\theta^i_t \omega^i) = t_*(\omega^i) - t$.
2. the stopping time $S^{i\prime}$ with the survivor function (3) has the memoryless property because

$$\begin{aligned} P_{x^i}(S^{i\prime} > t + s | S^{i\prime} > t) &= \frac{P_{x^i}\{\omega^i | m^i(\omega^i) > \Lambda^i_{t+s}(\omega^i)\}}{P_{x^i}\{\omega^i | m^i(\omega^i) > \Lambda^i_t(\omega^i)\}} \\ &= \frac{P_{x^i}\{\omega^i | m^i(\omega^i) > \Lambda^i_t(\omega^i) + \Lambda^i_s(\theta^i_t \omega^i)\}}{P_{x^i}\{\omega^i | m^i(\omega^i) > \Lambda^i_t(\omega^i)\}} \\ &= P_{x^i_t}\{\omega^i | m^i(\omega^i) > \Lambda^i_s(\theta^i_t \omega^i)\} \\ &= P_{x^i_t}(S^{i\prime} > s) \end{aligned}$$

   (we have used the fact that m^i has the memoryless property, being an exponentially distributed random variable, and the additivity of Λ^i_t w.r.t. t since this is an additive functional).

Since, for each $i \in Q$, the stopping time S^i is the infimum of t^* and $S^{i'}$, the two above facts easily implies the 'memoryless' property of S^i (it is easy to prove that the infimum of two memoryless stopping times is still a memoryless stopping time).
Thus, H is a Markov string obtained by mixing diffusion processes. Therefore, it inherits the strong Markov property from the component diffusions. ■

Corollary 8 *Any General Stochastic Hybrid Model H, under the standard assumptions of section* **??**, *is a Borel right process.*

Proof. The statement of the corollary is immediate, since the state space is a Lusin space and H is a right process. ■

As we discusses in the context of Markov strings, a GSHS might be thought of as a 'restriction' of a random evolution process [Sie81], whose components are diffusion processes defined on different state spaces. We can consider each diffusion component evolving on $\overline{X}$. The first difference is that while a GSHS is defined only on $\bigcup_{i\in Q}\{i\}\times X^i$ a random evolution process should be defined on the entire product space $Q \times \overline{X}$. The second difference is that whilst for a random evolution process the jump times from one process to another are driven only by transition rates, for a GSHS these might be also boundary hitting times of modes.

However, contrary to [Sie81], GSHS are not always standard processes as the random evolution processes.

The Process Generator. We denote by $\mathcal{B}_b(X)$ the set of all bounded measurable functions $f : X \to \mathbb{R}$. This is a Banach space under the norm $\|f\| = \sup_{x\in X}|f(x)|$. Associated with the semigroup (P_t) is its *strong generator* which is the 'derivative' of P_t at $t = 0$. Let $D(L) \subset \mathcal{B}_b(X)$ be the set of functions f for which the following limit exists $\lim_{t\searrow 0}\frac{1}{t}(P_tf - f)$ and denote this limit Lf. This refers to convergence in the norm $\|\cdot\|$, i.e. for $f \in D(L)$ we have $\lim_{t\searrow 0}||\frac{1}{t}(P_tf - f) - Lf|| = 0$. Specifying the domain $D(L)$ is an essential part of specifying L.

Proposition 9 (Martingale property) *[Dav93] For $f \in D(L)$ we define the real-valued process $(C^f_t)_{t\geq 0}$ by*

$$C^f_t = f(x_t) - f(x_0) - \int_0^t Lf(x_s)ds. \tag{27}$$

Then for any $x \in X$, the process $(C^f_t)_{t\geq 0}$ is a martingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, P_x)$.

There may be other functions f, not in $D(L)$, for which something akin to (27) is still true. In this way we get the notion of *extended generator* of the process.
Let $D(\widehat{L})$ be the set of measurable functions $f : X \to \mathbb{R}$ with the following property: there exists a measurable function $h : X \to \mathbb{R}$ such that $t \to h(x_t)$ is integrable $P_x - a.s.$ for each $x \in X$ and the process

$$C^f_t = f(x_t) - f(x_0) - \int_0^t h(x_s)ds$$

is a local martingale. Then we write $h = \widehat{L}f$ and call $(\widehat{L}, D(\widehat{L}))$ the extended generator of the process (x_t).
Following [Dav93], for $A \in \mathcal{B}(\overline{X})$ define p, p^* and $\widetilde{p}$ as follows:

$$p(t, A) = \sum_{k=1}^{\infty} I_{(t\geq T_k)} I_{(x_{T_k}\in A)};$$

$$p^*(t) = \sum_{k=1}^{\infty} I_{(t \geq T_k)} I_{(x_{T_k^-} \in \partial X)};$$

$$\widetilde{p}(t, A) = \int_0^t R(x_s, A)\lambda(x_s)ds + \int_0^t R(A, x_{s-})dp^*(s)$$

$$\widetilde{p}(t, A) = \sum_{T_k \leq t} R(x_{T_k-}, A).$$

Note that p, p^* are counting processes, $p^*(t)$ is counting the number of jumps from the boundary of the process (x_t). $\widetilde{p}(t, A)$ is the compensator of $p(t, A)$ (see [Dav93] for more explanations). The process $q(t, A) = p(t, A) - \widetilde{p}(t, A)$ is a local martingale.

Given a function $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ and a vector field $b : \mathbb{R}^n \to \mathbb{R}^n$, we use $\mathcal{L}_b f$ to denote the Lie derivative of f along b given by $\mathcal{L}_b f(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) f_i(x)$. Given a function $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$, we use $\mathbb{H}^f$ to denote the Hamiltonian operator applied to f, i.e. $\mathbb{H}^f(x) = (h_{ij}(x))_{i,j=1\ldots n} \in \mathbb{R}^{n \times n}$, where $h_{ij}(x) = \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$. A^T denotes the transpose matrix of a matrix $A = (a_{ij})_{i,j=1\ldots n} \in \mathbb{R}^{n \times m}$ and $Tr(A)$ denotes its trace.

Theorem 10 (GSHS generator) *Let H be an GSHS as in definition 1. Then the domain $D(L)$ of the extended generator L of H, as a Markov process, consists of those measurable functions f on $X \cup \partial X$ satisfying:*
1. $f : \overline{X} \to \mathbb{R}$, $\mathcal{B}-$measurable such that for each $i \in Q$ the restriction $f^i = f|_{X^i}$ is twice differentiable.
2. the boundary condition

$$f(x) = \int_{\mathbb{X}} f(y)R(x, dy), \; x \in \partial X;$$

3. $Bf \in L_1^{loc}(p)$ (see [2]) where

$$Bf(x, s, \omega) := f(x) - f(x_{s-}(\omega)).$$

For $f \in D(L)$, Lf is given by

$$Lf(x) = L_{cont} f(x) + \lambda(x) \int_{\overline{X}} (f(y) - f(x))R(x, dy) \tag{28}$$

where:

$$L_{cont} f(x) = \mathcal{L}_b f(x) + \frac{1}{2} Tr(\sigma(x)\sigma(x)^T \mathbb{H}^f(x)). \tag{29}$$

Proof. Let $(\widetilde{L}, D(\widetilde{L}))$ be the extended generator of (x_t). We want to show that $(\widetilde{L}, D(\widetilde{L})) = (L, D(L))$. Suppose first that f satisfies 1-3. Then $Bf \in L_1^{loc}(\widetilde{p})$ and $\int_{[0,t] \times \overline{X}} Bf d\widetilde{p} = I_1 + I_2$, where

$$\begin{aligned} I_1 &= \int_{[0,t]} \int_{\overline{X}} (f(y) - f(x_s))R(x_s, dy)\lambda(x_s)ds \\ I_2 &= \int_{[0,t]} \int_{\overline{X}} (f(y) - f(x_{s-}))R(x_{s-}, dy)dp^*(s). \end{aligned}$$

[2] Following [Dav93], f is in $L_1^{loc}(p)$ if for some sequence of stopping times $\sigma_n \uparrow \infty$

$$E_x \sum_i |f(x_{T_i \wedge \sigma_n}) - f(x_{T_i \wedge \sigma_n -})| < \infty$$

Now the support of p^* is contained in the countable set $\{s : x_{s-} \in \partial X\}$ and because of the boundary condition 2. the second integral I_2 vanishes. Thus

$$\int_{[0,t]\times\overline{X}} Bf dq = \sum_{T_k\leq t} (f(x_{T_k}) - f(x_{T_k-})) - \int_{[0,t]} \int_{\overline{X}} (f(y) - f(x_s))R(x_s, dy)\lambda(x_s)ds.$$

This is a local martingale because of condition 3. Let T_m denote the last jump time prior or equal to t. Then

$$\sum_{T_k\leq t} (f(x_{T_k}) - f(x_{T_k-})) = \{f(x_t) - f(x_{Tm})\} + S_m$$

where

$$\begin{aligned} S_m = &\sum_{k=1}^{m} (f(x_{T_k}) - f(x_{T_{k-1}}))\} - \{f(x_t) - f(x_{Tm}) + \\ &+ \sum_{k=1}^{m} (f(x_{T_k-}) - f(x_{T_{k-1}}))\}. \end{aligned}$$

The first bracketed term on the right is equal to $f(x_t) - f(x)$. Note that $x_{T_k-} = x^{i_{k-1}}_{T_k-T_{k-1}}$, if $x_{T_{k-1}} = (i_{k-1}, x^{i_{k-1}}_{k-1})$. Then Itô-formula gives the second term

$$f(x_{T_k-}) - f(x_{T_{k-1}}) = \int_{T_{k-1}}^{T_k} L_{cont}f(x_s)ds + \int_{T_{k-1}}^{T_k} < \sigma(x_s), \nabla f(x_s) > dW(s).$$

The second term is therefore equal to $\int_0^t L_{cont}f(x_s)ds + \int_0^t < \sigma(x_s), \nabla f(x_s) > dW(s)$ and we obtain

$$C_t^f := f(x_t) - f(x) - \int_0^t Lf(x_s)ds = \int_0^t < \sigma(x_s), \nabla f(x_s) > dW(s) + \int_{[0,t]\times\overline{X}} Bf dq$$

is a local martingale (the sum between a continuous martingale and a discrete martingale), where L is given by (28). Thus $f \in D(\widehat{L})$ and $\widehat{L}f = Lf$.
Conversely, suppose that $f \in D(\widehat{L})$. Then the process $M_t := f(x_t) - f(x) - \int_0^t h(x_s)ds$ is a local martingale, where $h = \widehat{L}f$. Then M_t must be the sum between a continuous martingale M_t^c and a discrete martingale M_t^d. From Th.(26.12), p.69 [Dav93], we have $M_t^d = M_t^\rho$ for some predictable integrand $\rho \in L_1^{loc}(p)$, where
$M_t^\rho = \int_{\overline{X}\times\mathbb{R}_+} \rho I_{(s\leq t)} dq = \sum_{T_k\leq t} \rho(x_{T_k}, T_k, \omega) - \int_0^t \int_{\overline{X}} \rho(y, s, \omega)\{R(x_s, dy)\lambda(x_s)ds - R(x_{s-}, dy)dp^*(s)\}$.
Since M_t^d and M_t^ρ agree, their jumps ΔM_t^d and ΔM_t^ρ must agree; these only occur when $t = T_k$ for some k and are given by: $\Delta M_t^d = f(x_t) - f(x_{t-})$; $\Delta M_t^\rho = \rho(x_t, t, \omega) - \int_{\overline{X}} \rho(y, t, \omega)R(x_{t-}, dy)I_{(x_{t-}\in\partial X)}$. Thus $\rho(x_t, t, \omega) = f(x_t) - f(x_{t-})$ on the set $(x_{t-} \notin \partial X)$, which implies that $\rho(x, t, \omega) = f(x) - f(x_{t-})$ for all (x, t) except perhaps a set to which the process 'never jumps', i.e. $G \subset \mathbb{R}_+ \times X$ such that $E_z \int_G p(dt, dx) = 0$, $\forall z \in X$.
Suppose that $z = x_{t-} \in \partial X$. Then equating ΔM_t^d and ΔM_t^ρ gives $f(x_t) - f(z) = \rho(x_t, t, \omega) - \int_{\overline{X}} \rho(y, t, \omega)R(z, dy)$ and hence $f(x) - f(z) = \rho(x, t, \omega) - \int_{\overline{X}} \rho(y, t, \omega)R(z, dy)$, except on a set $A \in \mathcal{B}(X)$ such that $R(z, A) = 0$. Integrating both sides of the previous equality with respect to $R(z, dx)$, we obtain $\int_{\overline{X}} f(x)R(z, dx) - f(z) = \int_{\overline{X}} \rho(x, t, \omega)R(z, dx) - \int_{\overline{X}} \rho(y, t, \omega)R(z, dy) = 0$.
Thus f satisfies the boundary condition. For fixed z, define $\widetilde{\rho}(x, t, \omega) = \rho(x, t, \omega) - (f(x) - f(z))$.

Using the boundary condition we get $\int_{\overline{X}} \widetilde{\rho}(y,t,\omega)R(z,dy) = \int_{\overline{X}} \rho(y,t,\omega)R(z,dy) = \widetilde{\rho}(x,t,\omega)$. Then $\widetilde{\rho}(x,t,\omega) = \int_{\overline{X}} \widetilde{\rho}(y,t,\omega)R(z,dy)$.
However, the right-hand side does not depend on x, and hence $\widetilde{\rho}(x,t,\omega) = u(t,\omega)$ for some predictable process u. The general expression for ρ is thus

$$\rho(x,t,\omega) = f(x) - f(x_{t-}) + u(t,\omega)I_{(x_{t-}\in\partial X)}.$$

Inserting this in the expression of M_t^ρ we find that M_t^ρ does not depend on u, then we can take $u \equiv 0$, obtaining $\rho = Bf$; hence the part 3 of theorem is satisfied.

Finally, consider the sample paths of M_t, $M_t^{Bf} + M_t^c$, for $t < T_1(\omega)$, starting at $x \in X$. We have

$$M_t = f(x_t(\omega^{i_0})) - f(x) + \int_0^t h(x_s(\omega^{i_0}))ds$$

while, because $p = p^* = 0$ on $[0, T_1)$,
$M_t^{Bf} = -\int_{[0,t]} \int_{\overline{X}} (f(y) - f(x_s(\omega^{i_0})))R(x_s(\omega^{i_0}), dy)\lambda(x_s(\omega^{i_0}))ds.$
So, since $M_t = M_t^{Bf} + M_t^c$ for all t a.s., it must be the case that $M_t = M_t^c$ for $t \in [0, T_1)$ and the generator coincides with the generator L_{cont} associated to the stochastic equation, the function $f(x_t(\omega^{i_0}))$ should have second order derivatives on $[0, T_1)$. The general case follows by concatenation. Similar calculations show that

$$M_t^{Bf} + M_t^c = f(x_t) - f(x) - \int_0^t Lf(x_s)ds, \forall t \geq 0$$

with L given by (28). Hence $f \in D(L)$ and $Lf = \widehat{L}f$. ■

7 Conclusions

7.1 Final Remarks

In this paper we set up the notion of Markov string, which is roughly speaking, a concatenation of Markov processes. This notion has arised as a result of our research on stochastic hybrid system modelling [HLS00, BL03, Buj04, PBLD03] and it aims to be a very general formalization of all existing models of stochastic hybrid systems. The Markov string concept has been proved to be a very powerful tool in the studying of the general models of stochastic hybrid processes GSHS introduced at the beginning of the paper.

One of the main contributions of this work is the proof of the strong Markov property. Since GSHS are a particular class of Markov strings, this property holds also for them.

In the end of this paper, based on the strong Markov property of GSHS we have developed the extended generator of this model.

7.2 Related work

A well-known and very powerful class of continuous time stochastic processes with stochastic jumps (for the discrete state and also for the continuous state) is the piecewise-deterministic Markov processes (PDMP), introduced in [Dav93], and applied to hybrid system modelling in [BL03]. The other modelling approaches are those presented in [HLS00] (stochastic hybrid systems abbreviated SHS), [BM00] (stochastic hybrid models abbreviated SHM), [GAM97, GB03] (switching diffusion processes, abbreviated SDP), [BGS99] (general switching diffusion processes abbreviated GSDP), see, also, [PBLD03] for quick presentation and comparisons. A very general formal model for stochastic hybrid systems is proposed in [Buj04], which extends the model from [HLS00], where the deterministic differential equations for the continuous flow are replaced by their stochastic counterparts, and the reset maps are generalized to (state-dependent) distributions that define the probability density of the state after a discrete transition. In this model transitions are always triggered by deterministic conditions (guards) on the state.

GSHS generalize PDMP allowing a stochastic evolution (diffusion process) between two consecutive jumps, while for PDMP the inter-jump motion is deterministic, according to a vector field. As well, GSHS might be thought of as a kind of extended SHS for which the transitions between modes are triggered by some stochastic event (boundary hitting time and transition rate). Moreover, GSHS generalise SDP permitting that also the continuous state to have discontinuities when the process jumps from one diffusion to another.

Another model for stochastic hybrid processes with hybrid jumps, which allows switching diffusions with jumps both in the discrete state and the continuous state, is developed in [Blo03]. It can be shown that the class of these models can be considered as a subclass of GSHS whose stochastic kernel, which gives the post jump locations, is chosen in an appropriate way such that the change of the discrete state at a jump depends on the pre jump location (continuous and discrete) and the change of the continuous state depends on the pre jump location and on the new discrete state.

7.3 Future Work

Further developments of our model will include three main tracks.

1. it is necessary a study of the reachability problem for GSHS. One possible approach in this direction is the introduction of a bisimulation concept for GSHS. Reachability analysis and model checking are much easier when a concept of bisimulation is available. The state space can be drastically abstracted in some cases. A robust and very general definition of bisimulation for GSHS has been proposed in [Buj05].

2. it is natural to generalize the results on dynamic programming, relaxed controls, control via discrete-time dynamic programming, non-smooth analysis, from PDMP to GSHS.

3. in many applications, stochastic hybrid systems are distributed and they do communicate. An extension of GSHS with parallelism and communication is started in [BB05]

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